NORTH-HOLLAND

A Whittaker Function of Matrix Argument

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ABSTRACT

Whittaker functions arise naturally in many topics in physical, biological, and social sciences, such as input-output situations in econometric problems, storage-consumption situations, growth-decay situations, when dealing with bilinear forms in random variables, and so on. An account of some of these is available in Mathai (1993). The matrix-variate analogue is considered in this article. A Whittaker function of matrix argument is defined, and several new results on this function are established in this paper, complementing the results given earlier by Mathai and Pederzoli (1996). Some of these generalize the corresponding results in the scalar variable case. © 1998 Elsevier Science Inc.

1. INTRODUCTION

This article is a contiuation of the paper Mathai and Pederzoli (1997). For the notation and terminology the reader is referred to that paper or to Mathai (1993). All the matrices appearing here are $p \times p$ real symmetric positive definite unless stated otherwise. Some bivariate analogues of Whittaker

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functions can be seen in Buschman and Gupta (1975), Exton (1976), and Srivastava and Karlsson (1985).

In order to enhance the readability, the definition of the Whittaker function will be repeated here. We define a Whittaker function $W_{\alpha,\beta}(A)$ of the matrix argument $A = A' > 0$ by using the following integral representation: For $A = A' > 0$, $\mathrm{Re}(\beta - \alpha) > (p-3)/4$, let

$$\int_{Z=Z'>0} |Z|^{\beta-\alpha-(p+1)/4}|I+Z|^{\alpha+\beta-(p+1)/4}\mathrm{e}^{-\operatorname{tr} AZ}\,\mathrm{d}Z$$

$$= |A|^{-\beta-(p+1)/4}\Gamma_p\left(\beta-\alpha+\frac{p+1}{4}\right)\mathrm{e}^{\frac{1}{2}\operatorname{tr} A}W_{\alpha,\beta}(A). \quad (1.1)$$

For certain parameter values one can write a Whittaker function coming from (1.1) as a linear combination of confluent hypergeometric functions when $p = 1$. For $p > 1$ such a representation is not available because

$$\int_{X=X'>0} \neq \int_{0<X=X'<I} + \int_{X\geqslant I}.$$

Hence for $p > 1$ a separate study of the properties of $W(\cdot)$ in (1.1) is needed. When $p = 1$, various properties of Whittaker and related functions are studied in Saxena (1964). We will establish a number of results by using (1.1).

2. SOME INTEGRALS INVOLVING WHITTAKER FUNCTIONS OF MATRIX ARGUMENT

THEOREM 1. *For* $\mathrm{Re}(\nu \pm \beta) > (p-3)/4$, $\mathrm{Re}(\beta - \alpha) > (p-3)/4$, $X = X' > 0$,

$$\int_{X>0} |X|^{\nu-(p+1)/2}\mathrm{e}^{-\frac{1}{2}\operatorname{tr} X}W_{\alpha,\beta}(X)\,\mathrm{d}X$$

$$= \frac{\Gamma_p\left(\frac{p+1}{4}+\nu+\beta\right)\Gamma_p\left(\frac{p+1}{4}+\nu-\beta\right)}{\Gamma_p\left(\frac{p+1}{2}+\nu-\alpha\right)}.$$

Proof. Substitute the integral representation of $W_{\alpha,\beta}(X)$ from (1.1) to obtain the following: For $\mathrm{Re}(\beta-\alpha)>(p-3)/4$ the

$$\text{left side} = \int_{X>0} |X|^{\nu-(p+1)/2} e^{-\frac{1}{2}\operatorname{tr} X} W_{\alpha,\beta}(X)\,\mathrm{d}X$$

$$= \frac{\int_{X>0} |X|^{\nu+\beta-(p+1)/4}}{\Gamma_p\left(\beta-\alpha+\frac{p+1}{4}\right)} e^{-\operatorname{tr} X} \int_{Z>0} |Z|^{\beta-\alpha-(p+1)/4}$$

$$\times |I+Z|^{\alpha+\beta-(p+1)/4} e^{-\operatorname{tr} XZ}\,\mathrm{d}Z\,\mathrm{d}X.$$

Integrating out X by using a real matrix-variate gamma and then Z by using a real matrix-variate type-2 beta, and then substituting back, the result follows. ■

For matrix-variate gamma and beta integrals in the real case see Mathai (1993) or Mathai and Pederzoli (1997).

THEOREM 2. *For* $A=A'>0$, $X=X'>0$, $\mathrm{Re}(\beta-\alpha)>(p-3)/4$, $\mathrm{Re}(\beta-\alpha-\mu)>(p-3)/4$, $\mathrm{Re}\,\mu>(p-1)/2$,

$$\int_{X>I} |X-I|^{\mu-(p+1)/2} |X|^{-\beta-(p+1)/4} e^{\frac{1}{2}\operatorname{tr} AX} W_{\alpha,\beta}(AX)\,\mathrm{d}X$$

$$= |A|^{-\mu/2}\Gamma_p(\mu) e^{\frac{1}{2}\operatorname{tr} A} W_{\alpha+\mu/2,\,\beta-\mu/2}(A) \frac{\Gamma_p\left(\beta-\alpha-\mu+\frac{p+1}{4}\right)}{\Gamma_p\left(\beta-\alpha+\frac{p+1}{4}\right)}.$$

Proof. Note that when A, B, C are $p\times p$ symmetric matrices one has $\mathrm{tr}(ABC)=\mathrm{tr}(BCA)=\mathrm{tr}(CAB)=\mathrm{tr}(ACB)$. Hence from (1.1) note that $W_{\alpha,\beta}(AB)=W_{\alpha,\beta}(BA)$. Thus in the trace we may permute the matrices according to convenience. Write the integral representation for $W_{\alpha,\beta}(AB)$

from (1.1) to get the

$$\text{left side} = \frac{|A|^{\beta+(p+1)/4}}{\Gamma_p\left(\beta - \alpha + \frac{p+1}{4}\right)} \int_{X>I} |X - I|^{\mu-(p+1)/2}$$

$$\times \int_{Z>0} |Z|^{\beta-\alpha-(p+1)/4} |I + Z|^{\alpha+\beta-(p+1)/4} e^{-\operatorname{tr}[(AX)Z]}\, dX\, dZ.$$

Put $Y = X - I$ and integrate out X. Then for Re $\mu > (p-1)/2$,

$$\int_{X>I} |X - I|^{\mu-(p+1)/2} e^{-\operatorname{tr}(AXZ)}\, dX = e^{-\operatorname{tr} AZ} \int_{Y>0} |Y|^{\mu-(p+1)/2} e^{-\operatorname{tr}(AZY)}\, dY$$

$$= e^{-\operatorname{tr} AZ}\Gamma_p(\mu)|AZ|^{-\mu}.$$

Now the Z-integral, denoted by h, is the following:

$$h = \int_{Z>0} |Z|^{\beta-\alpha-\mu-(p+1)/4} |I + Z|^{\alpha+\beta-(p+1)/4} e^{-\operatorname{tr} AZ}\, dZ$$

$$= \int_{Z>0} |Z|^{(\beta-\mu/2)-(\alpha+\mu/2)-(p+1)/4}$$

$$\times |I + Z|^{(\alpha+\mu/2)+(\beta-\mu/2)-(p+1)/4} e^{-\operatorname{tr} AZ}\, dZ$$

$$= |A|^{-\beta+\mu/2-(p+1)/4} \Gamma_p\left(\beta - \alpha - \mu + \frac{p+1}{4}\right) e^{\frac{1}{2}\operatorname{tr} A} W_{\alpha+\mu/2,\, \beta-\mu/2}(A).$$

Now, substituting back, the result follows. ■

Note that, in Theorem 2, if the exponent of $|X|$ on the left is $\beta - (p+1)/4$, that is, $-\beta$ is replaced by β, then the result will be different. In this case when $W_{\alpha,\beta}(AX)$ is substituted on the left side we still have a factor $|X|^{2\beta}$. Thus the Y-integral will be of the form

$$\int_{Y>0} |Y|^{\mu-(p+1)/2} |I + Y|^{2\beta} e^{-\operatorname{tr}[AZ(I+Y)]}\, dY.$$

Set $U = (I + Y)^{1/2} Z (I + Y)^{1/2}$ for fixed Y. Then the Y-integral reduces to the form

$$\int_{Y>0} |Y|^{\mu-(p+1)/2} \left| I + (I+U)^{-1} Y \right|^{-((p+1)/4-\alpha-\beta)} \mathrm{d}Y$$
$$= |I+U|^{\mu} \Gamma_p(\mu) \frac{\Gamma_p\left(\frac{p+1}{4} - \alpha - \beta - \mu\right)}{\Gamma_p\left(\frac{p+1}{4} - \alpha - \beta\right)}$$

for $\operatorname{Re} \mu > (p-1)/2$, $\operatorname{Re}(-\alpha-\beta-\mu) > (p-3)/4$, evaluating it by using a type-2 matrix-variate beta integral. Now the U-integral can be interpreted in terms of a $W_{\alpha+\mu/2,\, \beta+\mu/2}(A)$. This result is stated as the next theorem.

THEOREM 3. *For* $A = A' > 0$, $X = X' > 0$, $\operatorname{Re} \mu > (p-1)/2$, $\operatorname{Re}(\alpha + \beta + \mu) < (p-3)/4$, $\operatorname{Re}(\beta - \alpha) > (p-3)/4$,

$$\int_{X>I} |X - I|^{\mu-(p+1)/2} |X|^{\beta-(p+1)/4} \mathrm{e}^{\frac{1}{2}\operatorname{tr}(AX)} W_{\alpha,\beta}(AX)\, \mathrm{d}X$$
$$= |A|^{-\mu/2} \Gamma_p(\mu) \frac{\Gamma_p\left(\frac{p+1}{4} - \alpha - \beta - \mu\right)}{\Gamma_p\left(\frac{p+1}{4} - \alpha - \beta\right)} W_{\alpha+\mu/2,\, \beta+\mu/2}(A).$$

THEOREM 4. *For* $\delta > 0$, $\operatorname{Re}(\gamma \pm \beta) > (p-3)/4$, $\operatorname{Re}(\beta - \alpha) > (p-3)/4$,

$$\int_{X>0} |X|^{\gamma-(p+1)/2} \mathrm{e}^{-(\delta+\frac{1}{2})\operatorname{tr} X} W_{\alpha,\beta}(X)\, \mathrm{d}X$$
$$= (\delta+1)^{-p[(p+1)/4+\gamma+\beta]} \frac{\Gamma_p\left(\gamma + \beta + \frac{p+1}{4}\right) \Gamma_p\left(\gamma - \beta + \frac{p+1}{4}\right)}{\Gamma_p\left(\gamma - \alpha + \frac{p+1}{2}\right)}$$
$$\times {}_2F_1\left(\frac{p+1}{4} + \beta - \alpha, \frac{p+1}{4} + \gamma + \beta;\right.$$
$$\left. \frac{p+1}{2} + \gamma - \alpha; \frac{\delta}{\delta+1} I\right).$$

Proof. Substitute $W_{\alpha,\beta}(X)$ from (1.1) to obtain the

$$\text{left side} = \int_{X>0} \frac{|X|^{\gamma+\beta-(p+1)/4} e^{-(\delta+1)\mathrm{tr}(X)}}{\Gamma_p\left(\beta-\alpha+\frac{p+1}{4}\right)}$$

$$\times \int_{Z>0} |Z|^{\beta-\alpha-(p+1)/4} |I+Z|^{\alpha+\beta-(p+1)/4} e^{-\mathrm{tr}\, XZ}\, \mathrm{d}Z\, \mathrm{d}X.$$

Evaluate the X-integral by using a real matrix-variate gamma and consider the resulting Z-integral, denoted by h:

$$h = \int_{Z>0} |Z|^{\beta-\alpha-(p+1)/4} |I+Z|^{\alpha+\beta-(p+1)/4}$$

$$\times \left| I + \frac{1}{\delta+1} Z \right|^{-[(p+1)/4+\gamma+\beta]} \mathrm{d}Z.$$

Put $Z = (I-U)^{-1/2} U (I-U)^{-1/2} \Rightarrow \mathrm{d}Z = |I-U|^{-(p+1)} \mathrm{d}U$, $|Z| = |U|\,|I-U|^{-1}$, $|I+Z| = |I-U|^{-1}$, $0 < U < I$. Then

$$h = \int_{0<U<I} |U|^{\beta-\alpha-(p+1)/4} |I-U|^{\gamma-\beta-(p+1)/4}$$

$$\times \left| I - \frac{\delta}{\delta+1} U \right|^{-[(p+1)/4+\gamma+\beta]} \mathrm{d}U.$$

Writing this as a ${}_2F_1$ by using Mathai (1993, p. 179), we have

$$h = \frac{\Gamma_p\left(\frac{p+1}{4}+\beta-\alpha\right)\Gamma_p\left(\frac{p+1}{4}+\gamma-\beta\right)}{\Gamma_p\left(\frac{p+1}{2}+\gamma-\alpha\right)}$$

$$\times {}_2F_1\left(\frac{p+1}{4}+\beta-\alpha, \frac{p+1}{4}+\gamma+\beta; \frac{p+1}{2}+\gamma-\alpha; \frac{\delta}{\delta+1} I\right).$$

Substituting back, the result follows. ■

REMARK 1. If $\gamma = \alpha + 2\beta$ and $\delta = 1$, then in the scalar case (that is, when $p = 1$), the hypergeometric function has the argument $\delta/(\delta + 1) = \frac{1}{2}$, which factorizes into gamma products.

THEOREM 5. *For* $A = A' > 0$, $B = B' > 0$, $\operatorname{Re}(\gamma \pm \beta) > (p-3)/4$, $\operatorname{Re}(\beta - \alpha) > (p-3)/4$, $X = X' > 0$,

$$\int_{X>0} |X|^{\gamma-(p+1)/2} e^{-\operatorname{tr} AX} W_{\alpha,\beta}(BX)\, dX$$

$$= \left|A + \tfrac{1}{2}B\right|^{-[\gamma+\beta+(p+1)/4]}$$

$$\times \frac{\Gamma_p\left(\gamma + \beta + \frac{p+1}{4}\right)\Gamma_p\left(\gamma - \beta + \frac{p+1}{4}\right)}{\Gamma_p\left(\gamma - \alpha + \frac{p+1}{2}\right)}$$

$$\times {}_2F_1\left(\frac{p+1}{4} + \beta - \alpha, \frac{p+1}{4} + \gamma + \beta;\right.$$

$$\left.\frac{p+1}{2} + \gamma - \alpha; I - B^{1/2}\left(A + \tfrac{1}{2}B\right)^{-1} B^{1/2}\right) \quad (2.1)$$

for $0 < B^{1/2}(A + \frac{1}{2}B)^{-1}B^{1/2} < I$ *or* $2B^{-1/2}AB^{-1/2} > I$,

$$= |B|^{-[\gamma+\beta+(p+1)/4]} \frac{\Gamma_p\left(\frac{p+1}{4} + \gamma + \beta\right)\Gamma_p\left(\frac{p+1}{4} + \gamma - \beta\right)}{\Gamma_p\left(\frac{p+1}{2} + \gamma - \alpha\right)}$$

$$\times {}_2F_1\left(\frac{p+1}{4} + \gamma - \beta, \frac{p+1}{4} + \gamma + \beta; \frac{p+1}{2} + \gamma - \alpha; -C\right) \quad (2.2)$$

for $\|C\| < 1$, *where* $C = B^{-1/2}AB^{-1/2} - \frac{1}{2}I$ *and* $\|(\cdot)\|$ *denotes a norm of* $(\cdot)$. *A sufficient condition is that*

$$0 < B^{-1/2}AB^{-1/2} - \tfrac{1}{2}I < I \quad \text{or} \quad 0 < \tfrac{1}{2}I - B^{-1/2}AB^{-1/2} < I.$$

Proof. Substitute the integral representation of $W_{\alpha,\beta}(BX)$ to get the

$$\text{left side} = \int_{X>0} |X|^{\gamma-(p+1)/2} e^{-\operatorname{tr} AX} W_{\alpha,\beta}(BX)\,\mathrm{d}X$$

$$= \int_{X>0} \frac{|X|^{\gamma+\beta+(p+1)/4-(p+1)/2}}{\Gamma_p\left(\frac{p+1}{4}+\beta-\alpha\right)} e^{-\operatorname{tr} AX - \frac{1}{2}\operatorname{tr} BX}$$

$$\times \int_{Z>0} |Z|^{\beta-\alpha-(p+1)/4} |I+Z|^{\alpha+\beta-(p+1)/4} e^{-\operatorname{tr}(BXZ)}\,\mathrm{d}Z\,\mathrm{d}X.$$

Evaluate the X-integral by using a gamma integral to get

$$\int_{X>0} |X|^{\gamma+\beta+(p+1)/4-(p+1)/2} e^{-\operatorname{tr}\{[(A+\frac{1}{2}B)+BZ]X\}}\,\mathrm{d}X$$

$$= \Gamma_p\left(\gamma+\beta+\frac{p+1}{4}\right)\left|\left(A+\tfrac{1}{2}B\right)+BZ\right|^{-[\gamma+\beta+(p+1)/4]} \tag{2.3}$$

for $\operatorname{Re}(\gamma+\beta) > (p-3)/4$. The Z-integral, denoted by h, is given by

$$h = \int_{Z>0} |Z|^{\beta-\alpha-(p+1)/4} |I+Z|^{\alpha+\beta-(p+1)/4}$$

$$\times \left|\left(A+\tfrac{1}{2}B\right)+BZ\right|^{-[\gamma+\beta+(p+1)/4]}\mathrm{d}Z$$

$$= \left|A+\tfrac{1}{2}B\right|^{-[\gamma+\beta+(p+1)/4]} \int_{Z>0} |Z|^{\beta-\alpha-(p+1)/4} |I+Z|^{\alpha+\beta-(p+1)/4}$$

$$\times \left|I+B^{1/2}\left(A+\tfrac{1}{2}B\right)^{-1}B^{1/2}Z\right|^{-[\gamma+\beta+(p+1)/4]}\mathrm{d}Z.$$

Note that in the determinant we can also replace $(A+\frac{1}{2}B)^{-1}B$ by $B^{1/2}(A+\frac{1}{2}B)^{-1}B^{1/2}$. Put $Z=(I-U)^{-1/2}U(I-U)^{-1/2} \Rightarrow I+Z^{-1}=U^{-1} \Rightarrow |Z|^{-(p+1)}\mathrm{d}Z = |U|^{-(p+1)}\mathrm{d}U \Rightarrow \mathrm{d}Z = |I-U|^{-(p+1)}\mathrm{d}U$ and $0<U<I$. Then

$$h = \left|A+\tfrac{1}{2}B\right|^{-[\gamma+\beta+(p+1)/4]} \int_{0<U<I} |U|^{\beta-\alpha-(p+1)/4} |I-U|^{\gamma-\beta-(p+1)/4}$$

$$\times \left|I-\left[I-B^{1/2}\left(A+\tfrac{1}{2}B\right)^{-1}B^{1/2}\right]U\right|^{-[\gamma+\beta+(p+1)/4]}\mathrm{d}U.$$

Now, writing the integral as a ${}_2F_1$ by using Mathai (1993, p. 179), we have

$$h = \left|A + \tfrac{1}{2}B\right|^{-[\gamma+\beta+(p+1)/4]}$$

$$\times \frac{\Gamma_p\left(\beta - \alpha + \frac{p+1}{4}\right)\Gamma_p\left(\gamma - \beta + \frac{p+1}{4}\right)}{\Gamma_p\left(\gamma - \alpha + \frac{p+1}{2}\right)}$$

$$\times {}_2F_1\left(\frac{p+1}{4} + \beta - \alpha, \frac{p+1}{4} + \gamma + \beta;\right.$$

$$\left.\frac{p+1}{2} + \gamma - \alpha; I - B^{1/2}\left(A + \tfrac{1}{2}B\right)^{-1}B^{1/2}\right)$$

for $0 < I - B^{1/2}(A + \frac{1}{2}B)^{-1}B^{1/2} < I \Rightarrow 0 < B^{1/2}(A + \frac{1}{2}B)^{-1}B^{1/2} < I \Rightarrow 2B^{-1/2}AB^{-1/2} > I$. Substituting back, one gamma is canceled, and we get (2.1).

But note that

$$\left|\left(A + \tfrac{1}{2}B\right) + BZ\right| = \left|\left(A - \tfrac{1}{2}B\right) + B(I - Z)\right|$$

$$= |B|\,|I - Z|\left|I + \left(B^{-1/2}AB^{-1/2} - \tfrac{1}{2}I\right)(I + Z)^{-1}\right|.$$

Now the Z-integral, denoted by h, is given by

$$h = |B|^{-[\gamma+\beta+(p+1)/4]}\int_{Z>0} |Z|^{\beta-\alpha-(p+1)/4}|I + Z|^{\alpha-\gamma-(p+1)/2}$$

$$\times\left|I + \left(B^{-1/2}AB^{-1/2} - \tfrac{1}{2}I\right)(I + Z)^{-1}\right|^{-[\gamma+\beta+(p+1)/4]}\mathrm{d}Z.$$

Put $U = (I + Z)^{-1} \Rightarrow \mathrm{d}Z = |U|^{-(p+1)}\,\mathrm{d}U$, $|Z| = |U|^{-1}|I - U|$, $|I + Z| = |U|^{-1}$, $0 < U < I$. Then

$$h = |B|^{-[\gamma+\beta+(p+1)/4]}\int_{0<U<I} |U|^{\gamma-\beta-(p+1)/4}|I - U|^{\beta-\alpha-(p+1)/4}$$

$$\times |I + CU|^{-[\gamma+\beta+(p+1)/4]}\,\mathrm{d}U,$$

where $C = B^{-1/2}AB^{-1/2} - \frac{1}{2}I$. Writing this integral by using Mathai (1993, p. 179), we have

$$h = |B|^{-[\gamma+\beta+(p+1)/4]} \frac{\Gamma_p\left(\frac{p+1}{4} + \gamma - \beta\right)\Gamma_p\left(\frac{p+1}{4} + \beta - \alpha\right)}{\Gamma_p\left(\frac{p+1}{2} + \gamma - \alpha\right)}$$

$$\times {}_2F_1\left(\frac{p+1}{4} + \gamma - \beta, \gamma + \beta + \frac{p+1}{4}; \frac{p+1}{2} + \gamma - \alpha; -C\right)$$

for $\|C\| < 1$, $\mathrm{Re}(\gamma - \beta) > (p-3)/4$, $\mathrm{Re}(\beta - \alpha) > (p-3)/4$. Substituting back, we get (2.2), observing that one gamma is canceled. ■

THEOREM 6. *For $T = T' > 0$, $X = X' > 0$, $\mathrm{Re}\,\lambda > (p-1)/2$, $\mathrm{Re}(\beta - \alpha) > (p-3)/4$, $\mathrm{Re}(\alpha + \beta) < (p-3)/4$, $\mathrm{Re}(\alpha + \beta + \lambda) < (p-3)/4$,*

$$\int_{0<X<T} |X|^{-\alpha-\lambda-(p+1)/2}|T - X|^{\lambda-(p+1)/2} e^{\frac{1}{2}\mathrm{tr}\,X} W_{\alpha,\beta}(X)\,\mathrm{d}X$$

$$= \frac{\Gamma_p(\lambda)\Gamma_p\left(\frac{p+1}{4} - \alpha - \beta - \lambda\right)\Gamma_p\left(\frac{p+1}{4} + \beta - \alpha - \lambda\right)}{\Gamma_p\left(\frac{p+1}{4} + \beta - \alpha\right)\Gamma_p\left(\frac{p+1}{4} - \alpha - \beta\right)}$$

$$\times |T|^{-\alpha-(p+1)/2} e^{\frac{1}{2}\mathrm{tr}\,T} W_{\alpha+\lambda,\beta}(T).$$

Proof. Substitute the integral representation for $W_{\alpha,\beta}(X)$. Then the

$$\text{left side} = \int_{0<X<T} |X|^{-\alpha-\lambda-(p+1)/2}|T - X|^{\lambda-(p+1)/2}$$

$$\times e^{\frac{1}{2}\mathrm{tr}\,X} \frac{|X|^{\beta+(p+1)/4} e^{-\frac{1}{2}\mathrm{tr}\,X}}{\Gamma_p\left(\beta - \alpha + \frac{p+1}{4}\right)}$$

$$\times \int_{Z>0} |Z|^{\beta-\alpha-(p+1)/4}|I + Z|^{\alpha+\beta-(p+1)/4} e^{-\mathrm{tr}\,XZ}\,\mathrm{d}Z\,\mathrm{d}X$$

$$= \frac{1}{\Gamma_p\left(\dfrac{p+1}{4} + \beta - \alpha\right)}$$

$$\times \int_{0<X<T} |X|^{\beta-\alpha-\lambda-(p+1)/4} |T - X|^{\lambda-(p+1)/2}$$

$$\times \int_{Z>0} |Z|^{\beta-\alpha-(p+1)/4} |I + Z|^{\alpha+\beta-(p+1)/4} \mathrm{e}^{-\mathrm{tr}\, XZ}\, \mathrm{d}Z\, \mathrm{d}X.$$

Put $Z = X^{-1/2}UX^{-1/2} \Rightarrow \mathrm{d}Z = |X|^{-(p+1)/2}\,\mathrm{d}U$. The factors containing X are $|X|^{-\alpha-\beta-\lambda-(p+1)/4}|T - X|^{\lambda-(p+1)/2}|X + U|^{\alpha+\beta-(p+1)/4}$ and the factors containing U are $|U|^{\beta-\alpha-(p+1)/4}\mathrm{e}^{-\mathrm{tr}\, U}$. Take out $|T|$ from $|T - X|$, and change to $Y = T^{-1/2}XT^{-1/2} \Rightarrow \mathrm{d}X = |T|^{(p+1)/2}\,\mathrm{d}Y$, $0 < Y < I$. Then the

$$\text{left side} = \frac{|T|^{-(p+1)/2}}{\Gamma_p\left(\dfrac{p+1}{4} + \beta - \alpha\right)}$$

$$\times \int_{0<Y<I} \int_{U>0} |Y|^{-\alpha-\beta-\lambda-(p+1)/4} |I - Y|^{\lambda-(p+1)/2}$$

$$\times |U|^{\beta-\alpha-(p+1)/4} |Y + T^{-1/2}UT^{-1/2}|^{\alpha+\beta-(p+1)/4} \mathrm{e}^{-\mathrm{tr}\, U}\, \mathrm{d}U.$$

Change to $V = T^{-1/2}UT^{-1/2} \Rightarrow \mathrm{d}U = |T|^{(p+1)/2}\,\mathrm{d}V$. Now the

$$\text{left side} = \frac{|T|^{\beta-\alpha-(p+1)/4}}{\Gamma_p\left(\dfrac{p+1}{4} + \beta - \alpha\right)}$$

$$\times \int_{0<Y<I} \int_{V>0} |Y|^{-\alpha-\beta-\lambda-(p+1)/4} |I - Y|^{\lambda-(p+1)/2}$$

$$\times |V|^{\beta-\alpha-(p+1)/4} |Y + V|^{\alpha+\beta-(p+1)/4} \mathrm{e}^{-\mathrm{tr}\, TV}\, \mathrm{d}V\, \mathrm{d}Y.$$

Put $Y = (I + W)^{-1} \Rightarrow dY = |I + W|^{-(p+1)} dW$, $W > 0$. Then the

$$\text{left side} = \frac{|T|^{\beta-\alpha-(p+1)/4}}{\Gamma_p\left(\frac{p+1}{4} + \beta - \alpha\right)}$$

$$\times \int_{W>0}\int_{V>0} |W|^{\lambda-(p+1)/2}$$

$$\times |I + (I + W)V|^{\alpha+\beta-(p+1)/4}|V|^{\beta-\alpha-(p+1)/4} e^{-\operatorname{tr} TV} dV\, dW.$$

Write $|I + (I + W)V| = |I + V||I + (I + V^{-1})^{-1}W|$. Integrate out W by using a type-2 beta integral after making the change $(I + V^{-1})^{-1/2}W(I + V^{-1})^{-1/2} = G$ for fixed V, to get

$$\int_{W>0} |W|^{\lambda-(p+1)/2}\left|I + \left(I + V^{-1}\right)^{-1}W\right|^{\alpha+\beta-(p+1)/4} dW$$

$$= |I + V^{-1}|^{\lambda} \frac{\Gamma_p(\lambda)\Gamma_p\left(\frac{p+1}{4} - \alpha - \beta - \lambda\right)}{\Gamma_p\left(\frac{p+1}{4} - \alpha - \beta\right)}$$

for $\operatorname{Re}\lambda > (p-1)/2$, $\operatorname{Re}[(p+1)/4 - \alpha - \beta - \lambda] > (p-1)/2$. Now the

$$\text{left side} = |T|^{-\alpha-(p+1)/2+[\beta+(p+1)/4]}$$

$$\times \frac{\Gamma_p(\lambda)\Gamma_p\left(\frac{p+1}{4} - \alpha - \beta - \lambda\right)}{\Gamma_p\left(\frac{p+1}{4} + \beta - \alpha\right)\Gamma_p\left(\frac{p+1}{4} - \alpha - \beta\right)}$$

$$\times \int_{V>0} |V|^{\beta-\alpha-\lambda-(p+1)/4}|I + V|^{\alpha+\beta+\lambda-(p+1)/4} e^{-\operatorname{tr} TV} dV.$$

But the V-integral can be written in terms of a Whittaker function. That is,

$$\int_{V>0} |V|^{\beta-\alpha-\lambda-(p+1)/4}|I+V|^{\alpha+\beta+\lambda-(p+1)/4} e^{-\operatorname{tr} TV} \, \mathrm{d}V$$

$$= |T|^{-\beta-(p+1)/4} \Gamma_p\left(\frac{p+1}{4}+\beta-\alpha-\lambda\right) e^{\frac{1}{2}\operatorname{tr} T} W_{\alpha+\lambda,\beta}(T).$$

Substituting back, the result follows. ■

Theorems 1, 3, 4, 6 generalize the formulae 7.621(11), 7.623(5), 7.621(10), and 7.623(4) of Gradshteyn and Ryshik (1980). In 7.623(4) the factor $e^{t/2}$ seems to be missing there.

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